Solution 6

Supplementary Problems

1. Find the volume of the ball $x^2+y^2+z^2+w^2 \le R^2$ in $\mathbb{R}^4$ by the formula

$$\text{vol} = \int_{-R}^{R} |B_w|\, dw \ ,$$

where $|B_w|$ is the volume of the cross section of the ball at height w. The answer is $\pi^2 R^4/2$.

Solution. The cross section is a three dimensional ball of radius $\sqrt{R^2-w^2}$. Using this formula, the volume of the four dimensional ball is

$$\begin{aligned}
\int_{-R}^{R} |B_w|\, dw &= 2\int_0^R |B_w|\, dw \\
&= 2\int_0^R \frac{4\pi}{3}(R^2-w^2)^{3/2}\, dw \\
&= \frac{8\pi}{3}R^4 \int_0^{\pi/2} \sin^4\theta\, d\theta \quad (w = R\cos\theta) \\
&= \frac{\pi^2}{2}R^4
\end{aligned}$$

2. Let D be a region in the plane which is unchanged under the map $(x,y) \mapsto (-x,-y)$. Show that

$$\iint_D f(x,y)\, dA(x,y) = 0 \ ,$$

when f is odd, that is, $f(-x,-y) = -f(x,y)$ in D. This problem has appeared in a previous exercise. Now you are asked to apply the change of variables formula in two dimension.

Solution. Just because the Jacobian of the map $(x,y) \mapsto (-x,-y)$ is equal to 1.

3. The rotation by an angle θ in anticlockwise direction is given by $(x,y) = (\cos\theta\ u - \sin\theta\ v, \sin\theta\ u + \cos\theta\ v)$. Verify that rotation leaves the area unchanged.

Solution. Let G be a region in the plane. The area of G is defined to be $\iint_G 1\, dA(u,v)$. After the rotation G to D, and the area of D is $\iint_D 1\, dA(x,y)$. The Jacobian of the change of variables $\dfrac{\partial(x,y)}{\partial(u,v)}$ is easily calculated to be 1. Therefore,

$$|D| = \iint_D 1\, dA(x,y) = \iint_G 1 \times 1\, dA(u,v) = |G| \ .$$

Note. It is easy to verify that other Euclidean motions such as translations and reflections also leave the area unchanged. Their Jacobians are all equal to 1 or -1.

4. Consider the map $(u,v) \mapsto (x,y) = (u^2, v)$ which maps the square $R_1 = [-1,1]\times[0,1]$ onto $R_2 = [0,1]\times[0,1]$. Show that in general

$$\iint_{R_2} f(x,y) dA(x,y) \neq \iint_{R_1} f(u^2,v) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA(u,v) \ .$$

(Hint: It suffices to take $f(x, y) \equiv 1$.) Why?

Solution. On one hand,

$$\iint_{R_2} dA(x, y) = \int_0^1 \int_0^1 dxdy = 1 \ .$$

On the other hand, the Jacobian determinant for this map is $2u$. Letting R_3 to be the part of R_1 on the right and R_4 the part on the left, both of which are unit squares.

$$\begin{aligned}\iint_{R_1} |2u| \, dA(u, v) &= \iint_{R_3} |2u| \, dA(u, v) + \iint_{R_4} |2u| \, dA(u, v) \\ &= \int_0^1 \int_0^1 2u \, dvdu + \int_{-1}^0 \int_0^1 (-2u) \, dvdu \\ &= 2 \ .\end{aligned}$$

The change of variables formula is not valid because this map is not one to one. In fact, it is two to one.

Note. Advanced results in this direction are the Area Formula and the Coarea Formula which apply to maps not necessarily one-to-one. Google for it.